Online Local Volatility Calibration by Convex Regularization with Morozov's Principle and Convergence Rates

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Abstract

In this article we address the regularization of the ill-posed problem of determining the local volatility surface (as a function of time to maturity and price) from market given option prices. We integrate the ever-increasing flow of option price information into the well-accepted local volatility model of Dupire. This leads to considering both the local volatility surfaces and their corresponding prices as indexed by the observed underlying stock price as time goes by in appropriate function spaces.

The parameter to data map consists of a nonlinear operator that maps the (variable) diffusion coefficient of a parabolic initial value problem into its solutions evaluated at certain sets. We tackle the inverse problem by convex regularization techniques in appropriate Bochner-Sobolev spaces.

As a preparation, we prove key regularity properties that enable us to apply convex regularization techniques. This forward framework is then used to build a calibration technique that combines online methods with convex Tikhonov regularization tools. Such procedure is used to solve the inverse problem of local volatility identification. As a result, we prove convergence rates with respect to noise and a corresponding Morozov discrepancy principle for the regularization parameter. We conclude by illustrating and validating the theoretical results by means of numerical tests with synthetic as well as real data.

Keywords: Local Volatility Calibration, Convex Regularization, Online Estimation, Morozov's Principle, Convergence Rates.

Introduction

A number of interesting problems in nonlinear analysis are motivated by questions from mathematical finance. Among those problems, the robust identification of the variable diffusion coefficient that appears in Dupire's local volatility model [7, 12] presents substantial difficulties for its nonlinearity and ill-posedness. In previous works tools from Convex Analysis and Inverse Problem theory have been used to address this problem. See [6] and references therein.

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In this work, we incorporate the fact that as time evolves more data is available for the identification of Dupire's volatility surface. Thus we develop an *online* approach to the ill-posed problem of the local volatility surface calibration. Such surface is characterized by a non-negative two-variable function $\sigma = \sigma(\tau, K)$ of the time to expiration τ and the strike price K.

In what follows, we consider that the local volatility surfaces are indexed by the observed underlying asset price S_0. The reason for that stems from the fact that if we try to use information of prices observed on different dates, there is no financial or economical reason for the volatility surface to stay exactly the same. Thus, in principle we may have different volatility surfaces, although such change may be small.

Let us quickly review the standard Black-Scholes setting and Dupire's local volatility model. Recall that an option or derivative is a contract whose value depends on the value of an underlying stock or index. Perhaps the most well known derivative is an European call option, where the holder has the right (but not the obligation) to buy the underlying at time $t = T$ for a strike value K. We shall denote the stochastic process defining such underlying $S(t) = S(t,\omega)$, where as usual we assume that it is an adapted stochastic process on a suitable filtered probability space $(\Omega, \mathscr{U}, \mathbb{F}, \widetilde{\mathbb{P}})$, where $\mathbb{F} = \{\mathbb{F}_t\}_{t\in\mathbb{R}}$ is a filtration [15].

It is well known [7, 12, 15] that the value C of an European call option with strike K and expiration $T = t + \tau$, where t is the current time, satisfies:

$$\begin{cases} -\dfrac{\partial C}{\partial \tau} + \dfrac{1}{2}\sigma^2(\tau,K)K^2\dfrac{\partial^2 C}{\partial K^2} - bK\dfrac{\partial C}{\partial K} & = & 0 & \tau > 0,\ K \geq 0 \\ C(\tau = 0, K) & = & (S_0 - K)^+, & \text{for } K > 0, \\ \lim\limits_{K\to+\infty} C(\tau,K) & = & 0, & \text{for } \ \tau > 0, \\ \lim\limits_{K\to 0^+} C(\tau,K) & = & S_0, & \text{for } \ \tau > 0 \end{cases} \tag{1}$$

where b is the difference between the continuously compounded interest and dividend rates of the underlying asset. In what follows, we assume that such quantities are constant. Defining the diffusion parameter $a(\tau,K) = \sigma(\tau,K)^2/2$, Problem (1) leads to the following parameter to solution map:

$$\begin{array}{rcl} F : D(F) \subset X & \longrightarrow & Y \\ a \in D(F) & \longmapsto & F(a) = C \in Y \end{array}$$

where X and Y are Hilbert spaces to be properly defined below. $D(F)$ is the domain of the parameter to solution map (not necessarily dense in X) and $C = C(a, \tau, K)$ is the solution of Problem (1) with diffusion parameter a.

The inverse problem of local volatility calibration, as it was tackled in previous works [4, 5, 6, 8], consists in given option prices C, find an element $\tilde{a}$ of $D(F)$ such that $F(\tilde{a}) = C$. The operator F is compact and weakly closed. Thus, this inverse problem is ill-posed. In [4, 5, 6, 8] different aspects of Tikhonov regularization were analyzed. It is characterized by the following: Find an element of

$$\operatorname{argmin}\left\{\|F(a) - C\|_Y^2 + \alpha f_{a_0}(a)\right\} \quad \text{subject to} \quad a \in D(F) \subset X,$$

where f_{a_0} is a weak lower semi-continuous convex coercive functional. The analysis presented in [4, 5, 6, 8] was based on an *a priori* choice of the regularization parameter with convex regularization tools.

In contrast, in the present work we explore the dependence of the local volatility surface on the observed asset price in order to incorporate different option price surfaces in the same

procedure of Tikhonov regularization. More precisely, we consider the map

$$\begin{array}{rcl} \mathcal{U} : D(\mathcal{U}) \subset \mathcal{X} & \longmapsto & \mathcal{Y} \\ \mathcal{A} \in D(\mathcal{U}) & \longmapsto & \mathcal{U}(\mathcal{A}) : S \in [S_{\min}, S_{\max}] \mapsto C(S, a(S)) \end{array}$$

where $C(S, \mathcal{A}(S))$ is the solution of (1) with $S_0 = S$ and $\sigma^2/2 = a(S)$. Moreover, $\mathcal{A}$ maps $S \in [S_{\min}, S_{\max}]$ to $a(S) \in D(F)$ in a well-behaved way.

In this context the inverse problem becomes the following: Given a family of option prices $\mathcal{C} \in \mathcal{Y}$, find $\widetilde{\mathcal{A}} \in D(\mathcal{U})$ such that $\mathcal{U}(\widetilde{\mathcal{A}}) = \mathcal{C}$. We shall see that the operator $\mathcal{U}$ is also compact and weakly closed. Thus, this problem is also ill-posed. The corresponding regularized problem is defined by the following:

Find an element of

$$\operatorname{argmin} \left\{ \int_{S_{\min}}^{S_{\max}} \|F(a(S)) - C(S)\|_Y^2 dS + \alpha f_{\mathcal{A}_0}(\mathcal{A}) \right\} \quad \text{subject to } \ \mathcal{A} \in D(\mathcal{U}).$$

The main contributions of the current work are the following:

Firstly, we extend the local volatility calibration problem to local volatility families. This new setting allows incorporating more data into the calibration problem, leading to an online Tikhonov regularization. We prove that the so-called direct problem is well-posed, i.e., the forward operator satisfies key regularity properties. This framework generalizes in a nontrivial way the structure used in previous works [4, 5, 6, 8] since it requires the introduction of more tools, in particular that of Bochner spaces.

Secondly, in this setting, we develop a convergence analysis in a general context, based on convex regularization tools. See [19].

Thirdly, we establish a relaxed version of Morozov's discrepancy principle with convergence rates. This allows us to find the regularization parameter appropriately for the present problem. See [2, 17].

The paper is divided as follows:

In Section 1, we present the setting of the direct problem. In Section 2, we define properly the forward operator and prove some key regularity properties that are important in the analysis of the inverse problem. This is done in Theorem 1 and Propositions 4, 5, 6 and 7. In Section 3, we tie up the inverse problem with convex Tikhonov regularization under an *a priori* choice of the regularization parameter. This is done in Theorems 2, 3, 4 and 5. In Section 4 we establish the Morozov discrepancy principle for the present problem with convergence rates. This is done in Theorems 7 and 8. Illustrative numerical tests are presented in Section 5.

1 Preliminaries

We start by setting the so-called direct problem. It is based on the pricing of European call options by a generalization of Black-Scholes-Merton model.

Performing the change of variables $y := \log(K/S_0)$ and $\tau := T - t$ on Problem (1) and defining

$$u(S_0, \tau, y) := C(S_0, \tau + t, S_0 \mathrm{e}^y) \text{ and } a(S_0, \tau, y) := \frac{1}{2}\sigma^2(S_0, \tau + t, S_0 \mathrm{e}^y),$$

it follows that $u(S_0,\tau,y)$ satisfies

$$\begin{cases} -\dfrac{\partial u}{\partial \tau} + a(S_0,\tau,y)\left(\dfrac{\partial^2 u}{\partial y^2} - \dfrac{\partial u}{\partial y}\right) + b\dfrac{\partial u}{\partial y} & = & 0 & \tau > 0, \ y \in \mathbb{R} \\ u(\tau = 0, y) & = & S_0(1-e^y)^+, & \text{for } \ y \in \mathbb{R}, \\ \lim\limits_{y\to+\infty} u(\tau,y) & = & 0, & \text{for } \ \tau > 0, \\ \lim\limits_{y\to-\infty} u(\tau,y) & = & S_0, & \text{for } \ \tau > 0. \end{cases} \tag{2}$$

Note that, σ and a are assumed strictly positive and are related by a smooth bijection (since $\sigma > 0$). Thus, in what follows we shall work only with the local variance a instead of volatility σ. This simplifies the direct and inverse problems analysis.

Let $I \subset \mathbb{R}$ be a possibly unbounded open interval, denote by $D = (0,T)\times I$ the set where problem (2) is defined. From [8] we know that (2) has a unique solution in $W^{1,2}_{2,loc}(D)$, the space of functions $u : (\tau, y) \in D \mapsto u(\tau, y) \in \mathbb{R}$ such that, it has locally squared integrable weak derivatives up to order one in τ and up to order two in y.

For latter purposes, we define the set where the diffusion parameter a lives. For fixed $\varepsilon > 0$, take scalar constants $a_1, a_2 \in \mathbb{R}$ such that $0 < a_1 \leq a_2 < +\infty$ and a fixed function $a_0 \in H^{1+\varepsilon}(D)$, with $a_0 < a < a_1$. Define

$$Q := \{a \in a_0 + H^{1+\varepsilon}(D) : a_1 \leq a \leq a_2\} \tag{3}$$

Note that Q is weakly closed and has nonempty interior under the standard topology of $H^{1+\varepsilon}(D)$. See the first two chapters of [5, 6] and references therein.

2 The Forward Operator

Since we assume that the local variance surface is dependent on the current price, we have to introduce proper spaces for the analysis of the problem. As it turns out, we have to make use of Bochner spaces techniques. For general results on the Bochner integral see [11, 18, 23]. The main reference for this section is [14].

We start with some definitions. Given a time interval, say $[0,\overline{T}]$, the realized prices $S(t)$ vary within $[S_{\min}, S_{\max}]$. For technical reasons we perform the change of variables $s = S(t) - S_{\min}$ and denote $S = S_{\max} - S_{\min}$. Thus $s \in [0, S]$. Hence, for each s, we denote $a(s) := a(s,\tau,y)$ the local variance surface correspondent to s.

Definition 1. *Given $\mathcal{A} \in L^2(0,S,H^{1+\varepsilon}(D))$, with $\mathcal{A} : s \mapsto a(s)$ (see [23]), we define its Fourier series $\hat{\mathcal{A}} = \{\hat{a}(k)\}_{k\in\mathbb{Z}}$ by*

$$\hat{a}(k) := \frac{1}{2S}\int_0^S a(s)\exp(-iks\pi/S)ds + \frac{1}{2S}\int_{-S}^0 a(-s)\exp(-iks\pi/S)ds.$$

It is well defined, since $\{s \mapsto a(s)\exp(-iks2\pi/S)\}$ is weakly measurable and

$$L^2(0,S,H^{1+\varepsilon}(D)) \subset L^1(0,S,H^{1+\varepsilon}(D))$$

by the Cauchy-Schwartz inequality.

Now we define a class of Bochner-type Sobolev spaces:

Definition 2. *Let $H^l(0,S,H^{1+\varepsilon}(D))$ be the space of $\mathcal{A} \in L^2(0,S,H^{1+\varepsilon}(D))$, such that*

$$\|\mathcal{A}\|_l := \sum_{k\in\mathbb{Z}} (1+|k|^l)^2 \|\hat{a}(k)\|^2_{H^{1+\varepsilon}(D)_{\mathbb{C}}} < \infty,$$

where $H^{1+\varepsilon}(D)_{\mathbb{C}} = H^{1+\varepsilon}(D) \oplus iH^{1+\varepsilon}(D)$ is the complexification of $H^{1+\varepsilon}(D)$. Moreover, $H^l(0,S,H^{1+\varepsilon}(D))$ is a Hilbert space with the inner product

$$\langle \mathcal{A}, \widetilde{\mathcal{A}} \rangle_l := \sum_{k\in\mathbb{Z}} (1+|k|^l)^2 \langle a(k), \tilde{a}(k) \rangle_{H^{1+\varepsilon}(D)_{\mathbb{C}}}.$$

Proposition 1. *[14, Lemma 3.2] For $l > 1/2$, each $\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D))$ has a continuous representative and the mapping $i_l : H^l(0,S,H^{1+\varepsilon}(D)) \hookrightarrow C(0,S,H^{1+\varepsilon}(D))$ is continuous (bounded). Moreover, we have the estimate*

$$\sup_{s\in[0,S]} \|u(s)\|_{H^{1+\varepsilon}(D)} \leq \|\mathcal{U}\|_l \left(2 \sum_{k=0}^{\infty} \frac{1}{(1+k^l)^2} \right)^{\frac{1}{2}}. \tag{4}$$

Defining $\langle \mathcal{A}, x \rangle_{H^{1+\varepsilon}(D)} := \{s \mapsto \langle a(s), x \rangle\}$ for $x \in H^{1+\varepsilon}(D)$ and $\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D))$, we have that $\langle \mathcal{A}, x \rangle_{H^{1+\varepsilon}(D)} \in H^l([0,S])$ and $\|\langle \mathcal{A}, x \rangle_{H^{1+\varepsilon}(D)}\|_{H^l([0,S])} \leq \|\mathcal{A}\|_l \|x\|_{H^{1+\varepsilon}(D)}$. Moreover, for every $\mathcal{A}, \mathcal{B} \in L^2(0,S,H^{1+\varepsilon}(D))$,

$$\langle \mathcal{A}, \mathcal{B} \rangle_{L^2(0,S,H^{1+\varepsilon}(D))} = \sum_{k\in\mathbb{Z}} \langle \hat{a}(k), \hat{b}(k) \rangle_{H^{1+\varepsilon}(D)_{\mathbb{C}}}.$$

In what follows, we shall need the following technical lemma:

Lemma 1. *Assume that $l > 1/2$. If the sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ converges weakly to $\widetilde{\mathcal{A}}$ in $H^l(0,S,H^{1+\varepsilon}(D))$, then, the sequence $\{a_k(s)\}_{k\in\mathbb{N}}$ weakly converge to $\tilde{a}(s)$ in $H^{1+\varepsilon}(D)$ for every $s \in [0,S]$.*

Proof: Take a $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ and $\widetilde{\mathcal{A}}$ as above. We want to show that, given a weak zero neighborhood U of $H^{1+\varepsilon}(D)$, then for a sufficiently large n, $a_n(s) - a(s) \in U$ for every $s \in [0,S]$. A weak zero neighborhood U of $H^{1+\varepsilon}(D)$ is defined by a set of $\alpha_1, ..., \alpha_K \in H^{1+\varepsilon}(D)$ and an $\epsilon > 0$ such that $g \in H^{1+\varepsilon}(D)$ is an element of U if $\max_{k=1,...,K} |\langle g, \alpha_n \rangle| < \epsilon$.

Since the immersion $H^l([0,S]) \hookrightarrow C([0,S])$ is compact and $H^l([0,S])$ is reflexive, it follows that each weak zero neighborhood of $H^l([0,S])$ is a zero neighborhood of $C([0,S])$. Furthermore, from Proposition 1 we know that $\langle \mathcal{A}, \alpha \rangle_{H^{1+\varepsilon}(D)} \in H^l([0,S])$ with its norm bounded by $\|\mathcal{A}\|_l \|\alpha\|_{H^{1+\varepsilon}(D)}$, for every $n \in \mathbb{N}$ and $\alpha \in H^{1+\varepsilon}(D)$. Thus, we take the smallest closed ball centered at zero, B, which contains $\langle \widetilde{\mathcal{A}}, \alpha_k \rangle_{H^{1+\varepsilon}(D)}$ with $k = 1, ..., K$ and every $\langle \mathcal{A}_n, \alpha_k \rangle_{H^{1+\varepsilon}(D)}$ with $n \in \mathbb{N}$ and $k = 1, ..., K$. Therefore, choosing $\epsilon > 0$ as above, it is true that for each $k = 1, ..., K$, there are $f_{k,1}, ..., f_{k,M(k)} \in H^l([0,S])$ and $\eta_k > 0$, such that $\|f\|_{C([0,S])} < \epsilon$ for every $f \in B$ with $\max_{m=1,...,M(k)} |\langle f, f_{k,m} \rangle| < \eta_k$. Hence, we define $\mathcal{C}_{k,m} := \alpha_k \otimes f_{k,m} \in H^l(0,S,H^{1+\varepsilon}(D))^*$ and the weak zero neighborhood $A = \cap_{k=1}^K A_k$ of $H^l(0,S,H^{1+\varepsilon}(D))$ with

$$A_k := \{\mathcal{A} \in H^l(0,S,H^{1+\varepsilon}(D)) \ : \ |\langle \mathcal{A}, \mathcal{C}_{k,m} \rangle| \leq \eta_k, \ m = 1, ..., M(k)\}.$$

As A is a weak zero neighborhood of $H^l(0,S,H^{1+\varepsilon}(D))$, it is true that for sufficiently large n, $\mathcal{A}_n - \widetilde{\mathcal{A}} \in A$, which implies that $a_n(s) - \tilde{a}(s) \in U$ for every $s \in [0,S]$, i.e., $\{a_n(s)\}_{n\in\mathbb{N}}$ weakly converges to $\tilde{a}(s)$ for every $s \in [0,S]$. □

Define:

$$\mathfrak{Q} := \{\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D)) : a(s) \in Q,\ \forall s \in [0, S]\}, \tag{5}$$

i.e., each $\mathcal{A}$ in $\mathfrak{Q}$ is the map $\mathcal{A} : s \in [0, S] \mapsto a(s) \in Q$. Note that $\mathfrak{Q}$ is the space of Q-valued paths, with Q defined in (3).

Proposition 2. *For $l > 1/2$, the set $\mathfrak{Q}$ is weakly closed and its interior is nonempty in $H^l(0, S, H^{1+\varepsilon}(D))$.*

Proof: By Lemma 1 and the fact that Q is weakly closed it follows that $\mathfrak{Q}$ is weakly closed. The assertion that the interior of $\mathfrak{Q}$ is nonempty follows from the fact that

$$H^l(0, S, H^{1+\varepsilon}(D)) \hookrightarrow C(0, S, H^{1+\varepsilon}(D))$$

is continuous and bounded (note that, given $\epsilon > 0$, $\widetilde{\mathcal{A}} = \{s \mapsto \tilde{a}(s)\}$ with $\underline{a}+\epsilon \le \tilde{a}(s) \le \overline{a}+\epsilon$ for every $s \in [0, S]$ is in the interior of $\mathfrak{Q}$). □

We stress that, in what follows, we always assume that $l > 1/2$, since it is enough to state our results concerning regularity aspects of the forward operator.

We define below the forward operator, that associates each family of local variance surfaces to the correspondent family of option price surfaces, determined by (2). Thus, for a given $a_0 \in Q$ we define:

$$\begin{array}{rcl} \mathcal{U} : \mathfrak{Q} & \longrightarrow & L^2(0, S, W_2^{1,2}(D)), \\ \mathcal{A} & \longmapsto & \mathcal{U}(\mathcal{A}) : s \in [0, S] \mapsto F(s, a(s)) \in W_2^{1,2}(D), \end{array}$$

where $[\mathcal{U}(\mathcal{A})](s) = F(s, a(s)) := u(s, a(s)) - u(s, a_0)$ and $u(s, a)$ is the solution of (2) with local variance a. The following results state some regularity properties concerning the forward operator.

Proposition 3. *The operator $F : [0, S] \times Q \longrightarrow W_2^{1,2}(D)$ is continuous and compact. Moreover, it is sequentially weakly continuous and weakly closed.*

We define below the concept of Frechét equi-differentiability for a family of operators.

Definition 3. *We call a family of operators $\{\mathcal{F}_s : Q \longrightarrow W_2^{1,2}(D) \,|\, s \in [0, S]\}$ Frechét equi-differentiable, if for all $\tilde{a} \in Q$ and $\epsilon > 0$, there is a $\delta > 0$, such that*

$$\sup_{s \in [0,S]} \|\mathcal{F}_t(\tilde{a} + h) - \mathcal{F}_s(\tilde{a}) - \mathcal{F}_s'(\tilde{a})h\| \le \epsilon \|h\|,$$

for $\|h\|_{H^{1+\varepsilon}(D)} < \delta$ and $\mathcal{F}_s'(\tilde{a})$ the Frechét derivative of $\mathcal{F}_s(\cdot)$ at $\tilde{a}$.

Thus, we have the following proposition.

Proposition 4. *The family of operators $\{F(s, \cdot) : Q \longrightarrow W_2^{1,2}(D) \,|\, s \in [0, S]\}$ is Frechét equi-differentiable.*

Proof: Given $\tilde{a} \in Q$ and $\epsilon > 0$, define $w = F(s, \tilde{a}+h) - F(s, \tilde{a}) - \partial_a F(s, \tilde{a})h$, it is equivalent to $w = u(s, \tilde{a} + h) - u(s, \tilde{a}) - \partial_a u(s, \tilde{a})h$. We denote $v := u(s, \tilde{a} + h) - u(s, \tilde{a})$. Thus, by linearity w satisfies

$$-w_\tau + \tilde{a}(w_{yy} - w_y) + bw_y = h(v_{yy} - v_y),$$

with homogeneous boundary condition. Such problem does not depend on s, as $\tilde{a}$ is independent of s. From the proof of Proposition 3 (see also [8]), we have

$$\|w\|_{W_2^{1,2}(D)} \le C \|h\|_{L^2(D)} \|v\|_{W_2^{1,2}(D)}$$

By the continuity of the operator F, given $\epsilon > 0$ we can chose $h \in H^{1+\varepsilon}(D)$ with

$$\|h\|_{H^{1+\varepsilon}(D)} \leq \delta$$

, such that $\|v\|_{W_2^{1,2}(D)} \leq \epsilon/C$ and thus the assertion follows. $\square$

The following theorem is the principal result of this section, since it states some properties that are at the core of the inverse problems analysis [10, 19]. For its proof see Appendix A.3.

Theorem 1. *The forward operator $\mathcal{U} : \mathfrak{Q} \longrightarrow L^2(0, S, W_2^{1,2}(D))$ is well defined, continuous and compact. Moreover, it is sequentially weakly continuous and weakly closed.*

The next results state necessary conditions for the convergence analysis. See [10, 19]. Its proof is in the Appendix A.3.

Proposition 5. *The operator $\mathcal{U}(\cdot)$ admits a one sided derivative at $\widetilde{\mathcal{A}} \in \mathfrak{Q}$ in the direction $\mathcal{H}$, such that $\widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$. The derivative $\mathcal{U}'(\widetilde{\mathcal{A}})$ satisfies*

$$\left\|\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}\right\|_{L^2(0,S,W_2^{1,2}(D))} \leq c\|\mathcal{H}\|_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

Moreover, $\mathcal{U}'(\widetilde{\mathcal{A}})$ satisfies the Lipschitz condition

$$\left\|\mathcal{U}'(\widetilde{\mathcal{A}}) - \mathcal{U}'(\widetilde{\mathcal{A}} + \mathcal{H})\right\|_{\mathcal{L}\left(H^l(0,S,H^{1+\varepsilon}(D)),L^2(0,S,W_2^{1,2}(D))\right)} \leq \gamma\|\mathcal{H}\|_{H^l(0,S,H^{1+\varepsilon}(D))}$$

for all $\widetilde{\mathcal{A}}, \mathcal{H} \in \mathfrak{Q}$ such that $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$.

The following result is a consequence of the compactness of $\mathcal{U}(\cdot)$.

Proposition 6. *The Frechét derivative of the operator $\mathcal{U}(\cdot)$ is injective and compact.*

Proof: Take $\mathcal{H} \in \ker\left(\mathcal{U}'(\widetilde{\mathcal{A}})\right)$. Thus, from the proof of Proposition 5, we have

$$h(s) \cdot (u_{yy} - u_y) = 0.$$

However, for each t, $G = u_{yy} - u_y$ is the solution of

$$\begin{cases} \partial_\tau G = \dfrac{1}{2}\left(\partial^2_{yy} - \partial_y\right)(a(s)G + bG) \\ G\,|_{\tau=0} = \delta(y) \,, \end{cases}$$

i.e., G is the Green's function of the Cauchy problem above. Thus, $G > 0$ for every y,$\tau > 0$ and $s \in [0, S]$. Therefore $h(t) = 0$. Since this holds for every $s \in [0, S]$, then the result follows. $\square$

We now make use of the bounded embedding of $L^2(0, S, W_2^{1,2}(D))$ into $L^2(0, S, L^2(D))$, since it implies that $\mathcal{U}$ satisfies the same results presented above with $L^2(0, S, L^2(D))$ instead of $L^2(0, S, W_2^{1,2}(D))$. Thus, we characterize the range of $\mathcal{U}'(\mathcal{A})$ as a subset of $L^2(0, S, L^2(D))$ and the range of $\mathcal{U}'(\mathcal{A})^*$ as a subset of $H^l(0, S, H^{1+\varepsilon}(D))$ in order to proceed in Section 3 the convergence analysis.

Proposition 7. *The operator $\mathcal{U}'(\mathcal{A}^\dagger)^*$ has a trivial kernel.*

Proof: For simplicity take $b = 0$. Denote

$$\mathcal{L} := -\partial_\tau + a(\partial yy - \partial_y)$$

the parabolic operator of Equation (2) with homogeneous boundary condition and $\mathcal{G}_{u_{yy}-u_y}$ the multiplication operator by $u_{yy} - u_y$. Thus, for each $s \in [0, S]$, we have $\partial_a u(s, \tilde{a}(s)) = \mathcal{L}^{-1}\mathcal{G}_{u_{yy}-u_y}$, where $\mathcal{L}^{-1}$ is the left inverse of $\mathcal{L}$ with null boundary conditions. By definition of

$$\mathcal{U}'(\widetilde{\mathcal{A}})^* : L^2(0, S, L^2(D)) \longrightarrow H^l(0, S, H^{1+\varepsilon}(D)),$$

we have,

$$\left\langle \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}, \mathcal{Z} \right\rangle_{L^2(0,S,L^2(D))} = \langle \mathcal{H}, \Phi \rangle_{H^l(0,S,H^{1+\varepsilon}(D))},$$

$\forall\ \mathcal{H} \in H^l(0, S, H^{1+\varepsilon}(D))$ and $\forall\ \mathcal{Z} \in L^2(0, S, L^2(D))$, with $\Phi = \mathcal{U}'(\widetilde{\mathcal{A}})^*\mathcal{Z}$. Thus, given any $\mathcal{Z} \in \ker\left(\mathcal{U}'(\widetilde{\mathcal{A}})^*\right)$, it follows that

$$\begin{aligned}
0 &= \left\langle \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}, \mathcal{Z} \right\rangle_{L^2(0,S,L^2(D))} = \int_0^S \left\langle \mathcal{L}^{-1}\mathcal{G}_{u_{yy}-u_y} h(s), z(s) \right\rangle_{L^2(D)} ds \\
&= \int_0^S \left\langle \mathcal{G}_{u_{yy}-u_y} h(s), [\mathcal{L}^{-1}]^* z(s) \right\rangle_{L^2(D)} ds = \int_0^S \left\langle \mathcal{G}_{u_{yy}-u_y} h(s), g(s) \right\rangle_{L^2(D)} ds
\end{aligned}$$

where g is a solution of the adjoint equation

$$g_\tau + (ag)_{yy} + (ag)_y = z$$

for each $s \in [0, S]$, with homogeneous boundary conditions. Since $z(t) \in L^2(D)$, we have that $g(s) \in H^{1+\varepsilon}(D)$ (see [16]) and $g \in L^2\left(0, S, H^{1+\varepsilon}(D)\right)$. Since $\mathcal{G} > 0$, from the proof of Proposition 6 and the fact that $h \in H^l(0, S, H^{1+\varepsilon}(D))$ is arbitrary, it follows that $g = 0$. Therefore $\mathcal{Z} = 0$ almost everywhere in $s \in [0, S]$. It yields that $\ker\left(\mathcal{U}'(a)^*\right) = \{0\}$. $\square$

Remark 1. *From the last proposition it follows that*

$$\ker\{\mathcal{U}'(\widetilde{\mathcal{A}})\} = \{0\} \Rightarrow \overline{\mathcal{R}\left\{\left(\mathcal{U}'(\widetilde{\mathcal{A}})\right)^*\right\}} = H^l(0, S, H^{1+\varepsilon}(D)).$$

In other words, the range of the adjoint operator of the Frechét derivative of the forward operator $\mathcal{U}$ at $\widetilde{\mathcal{A}}$ is dense in $H^l(0, S, H^{1+\varepsilon}(D))$.

To finish this section we shall present below the tangential cone condition for $\mathcal{U}$. It follows almost directly by the above results and Theorem 1.4.2 from [5].

Proposition 8. *The map $\mathcal{U}(\cdot)$ satisfies the local tangential cone condition*

$$\left\| \mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}}) - \mathcal{U}'(\widetilde{\mathcal{A}})(\mathcal{A} - \widetilde{\mathcal{A}}) \right\|_{L^2(0,S,W_2^{1,2}(D))} \leq \gamma \left\| \mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}}) \right\|_{L^2(0,S,W_2^{1,2}(D))} \quad (6)$$

for all $\mathcal{A}, \widetilde{\mathcal{A}}$ in a ball $B(\mathcal{A}^*, \rho) \subset \mathfrak{Q}$ with some $\rho > 0$ and $\gamma < 1/2$.

As a corollary we have the following result:

Corollary 1. *The operator $\mathcal{U}$ is injective.*

3 The Inverse Problem

Following the notation of Section 2, we want to define a precise and robust way of relating each family of European option price surfaces to the corresponding family of local volatility surfaces, both parameterized by the underlying stock price. We first present an analysis of existence and stability of regularized solutions, then we establish some convergence rates. We also prove Morozov's discrepancy principle for the present problem with the same convergence rates.

3.1 The Regularized Problem

The inverse problem of local volatility calibration can be restated as:
Given a family of of European call option price surfaces $\widetilde{\mathcal{U}} = \{s \mapsto \tilde{u}(s)\} \in L^2(0, S, L^2(D))$, *find the correspondent family of local variance surfaces* $\widetilde{\mathcal{A}} = \{s \mapsto \tilde{a}(s)\} \in \mathfrak{Q}$, *satisfying*

$$\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}}). \tag{7}$$

We call $\widetilde{\mathcal{U}}$ the observable variable and $\widetilde{\mathcal{A}}$ the unknown. This is an idealized situation since the model (7) has no uncertainties associated. Thus, to be more realistic, we assume that we can only observe corrupted data $\mathcal{U}^\delta$, satisfying a perturbed version of (7),

$$\mathcal{U}^\delta = \widetilde{\mathcal{U}} + \mathcal{E} = \mathcal{U}(\widetilde{\mathcal{A}}) + \mathcal{E} \tag{8}$$

where $\mathcal{E} = \{t \mapsto E(t)\}$ compiles all the uncertainties associated to this problem and $\widetilde{\mathcal{U}}$ is the unobservable noiseless data. We assume further that, the norm of $\mathcal{E}$ is bounded by the noise level $\delta > 0$. Moreover, for each $t \in [0, T]$, we assume that $\|E(t)\| \leq \delta/T$. These hypotheses imply that

$$\|\mathcal{U}^\delta - \widetilde{\mathcal{U}}\|_{L^2(0,S,L^2(D))} \leq \delta \quad \text{and} \quad \|u^\delta(t) - \tilde{u}(t)\|_{L^2(D)} \leq \delta/T \text{ for every } t \in [0, T]. \tag{9}$$

Proposition 1 gives that $\mathcal{U}(\cdot)$ is compact, implying that the associated inverse problem is ill-posed. It means that such inverse problem cannot be solved directly in a stable way. Hence, we must apply regularization techniques. This, roughly speaking, relies on stating the original problem under a more robust setting. More specifically, instead of looking for an $\mathcal{A}^\delta \in \mathfrak{Q}$ satisfying (8), we shall search for an $\mathcal{A}^\delta \in \mathfrak{Q}$ minimizing the following Tikhonov functional

$$\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha}(\mathcal{A}) = \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A})\|^2_{L^2(0,S,L^2(D))} + \alpha f_{\mathcal{A}_0}(\mathcal{A}). \tag{10}$$

The functional $f_{\mathcal{A}_0}$, which will be made more specific later, incorporates "*a priori*" information that we may have on $\mathcal{A}_0$ and has the goal of stabilizing the problem.

We shall see later that, under this setting, such minimizers are, in an appropriate sense, good approximations for the solution of (7). We recall that the forward operator is injective.

3.2 Some Properties of Minimizers

In order to guarantee the existence of stable minimizers for the functional (10), we assume that $f_{\mathcal{A}_0} : \mathfrak{Q} \to [0, \infty]$ is convex, coercive and weakly lower semi-continuous. A classical reference on convex analysis is [9]. Note that, these assumptions are not too restrictive, since they are fulfilled by a large class of functionals on $H^l(0, S, H^{1+\varepsilon}(D))$. A canonical

example is $f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}$, which is leads us to the classical Tikhonov regularization.

Recall that $\mathcal{U}$ is weakly continuous and $\mathfrak{Q}$ is weakly closed. Combining that with the required properties of $f_{\mathcal{A}_0}$, we have that

$$\mu_\alpha(M) = \left\{\mathcal{A} \in \mathfrak{Q} \,\middle|\, \mathcal{F}^\delta_{\mathcal{A}_0,\alpha}(\mathcal{A}) \leq M \right\}$$

are weakly pre-compact and the restriction of $\mathcal{U}$ to $\mu_\alpha(M)$ is weakly continuous.

As a consequence of this, we have the following three theorems. See [19].

Theorem 2 (Existence). *For every $\mathcal{U}^\delta \in L^2(0,S,L^2(D))$, there exists at least one element of $\mathfrak{Q}$ minimizing $\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha}(\cdot)$, the functional defined in (10).*

Before stating the next theorem, we need the following definition:

Definition 4 (Stability). *If $\widetilde{\mathcal{A}}$ is a minimizer of (10) with data $\mathcal{U}$, then it is called stable if for every sequence $\{\mathcal{U}_k\}_{k\in\mathbb{N}} \subset L^2(0,S,W^{1,2}_2(D))$ converging strongly to $\mathcal{U}$, the sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \subset \mathfrak{Q}$ of minimizers of $\mathcal{F}^{\mathcal{U}^k}_{\mathcal{A}_0,\alpha}(\cdot)$ has a subsequence converging weakly to $\widetilde{\mathcal{A}}$.*

Theorem 3 (Stability). *The minimizers of (10) are stable in the sense of Definition 4.*

The following result states that, when the noise level δ and the regularization parameter $\alpha = \alpha(\delta)$ vanish, then we can find a sequence of minimizers of (10) converging weakly to the solution of (7).

Theorem 4 (Convergence). *Assume that (7) has a solution and the map $\alpha : (0,\infty) \to (0,\infty)$, satisfies*

$$\lim_{\delta\to 0} \alpha(\delta) = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \frac{\delta^2}{\alpha(\delta)} = 0. \tag{11}$$

Moreover, we assume further that the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ converges to 0 and the elements of $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$, with $\mathcal{U}_k = \mathcal{U}^{\delta_k}$, satisfy $\|\tilde{\mathcal{U}} - \mathcal{U}_k\| \leq \delta_k$, with $\widetilde{\mathcal{U}}$ the noiseless data of (7). Denote $\alpha(\delta_k)$ by α_k for every $k \in \mathbb{N}$. Then, every sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of $\mathcal{F}^{\mathcal{U}_k}_{\mathcal{A}_0,\alpha_k}(\cdot)$, converges weakly to $\mathcal{A}^\dagger$, the unique solution of (7), with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$.

Therefore we can conclude that the present regularization procedure gives us reliable approximations for the solutions of Problem (7). Note that, such approximations depend on the magnitude of δ and thus on the choice of α and $f_{\mathcal{A}_0}$. Thus, Theorem 4 says the smaller δ is, if α is properly chosen, the less dependent on regularization the solutions are.

3.3 A Convergence Analysis

Making use of convex regularization tools, we provide some convergence rates with respect to the noise level. Thus, we need some abstract concepts, as the Bregman distance related to $f_{\mathcal{A}_0}$, q-coerciveness and the source condition related to operator $\mathcal{U}$. Such ideas were also used in [4, 5, 6, 8], but here they are extended to the context of online local volatility calibration. For the definitions of Bregman Distance and q-coerciveness see Appendix A.1.

In what follows we always assume that (7) has a (unique) solution which is an element of the Bregman domain $\mathcal{D}_B(f_{\mathcal{A}_0})$.

Before stating the result about convergence rates, we need the following two auxiliary lemmas. The first one introduces the so-called source condition. See [19].

Lemma 2. *For every $\xi^\dagger \in \partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$, there exists $\omega^\dagger \in L^2(0,S,L^2(D))$ and*

$$\mathscr{E} \in H^l(0,S,H^{1+\varepsilon}(D))$$

such that

$$\xi^\dagger = \left[\mathcal{U}'(\mathcal{A}^\dagger)\right]^* \omega^\dagger + \mathscr{E} \tag{12}$$

holds. Moreover, $\mathscr{E}$ can be taken such that $\|\mathscr{E}\|_{H^l(0,S,H^{1+\varepsilon}(D))}$ is arbitrarily small.

Lemma 2 follows from the fact that $\mathcal{R}(\mathcal{U}'(\mathcal{A}^\dagger)^*)$ is dense in $H^l(0,S,H^{1+\varepsilon}(D))$. See Proposition 7 in Section 2. Observe also that, we identify $L^2(0,S,L^2(D))^*$ and $H^l(0,S,H^{1+\varepsilon}(D))^*$ with $L^2(0,S,L^2(D))$ and $H^l(0,S,H^{1+\varepsilon}(D))$, respectively, since they are Hilbert spaces.

Now, the main result of this section.

Theorem 5 (Convergence Rates)**.** *Under the hypotheses of this section, let the map $\alpha : (0,\infty) \to (0,\infty)$ be such that $\alpha(\delta) \approx \delta$. Furthermore, assume that $f_{\mathcal{A}_0}(\cdot)$ is q-coercive with constant ζ, with respect to the norm of $H^l(0,S,H^{1+\varepsilon}(D))$. Then*

$$D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) = \mathcal{O}(\delta) \quad \text{ and } \quad \|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\| = \mathcal{O}(\delta).$$

Proof: Let $\mathcal{A}^\dagger$ and $\mathcal{A}^\delta_\alpha$ denote the solution of (7) and the minimizer of (10). It follows that,

$$\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}^\delta_\alpha) \le \|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^\delta\|^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \le \delta^2 + \alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger).$$

Since, $D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) = f_{\mathcal{A}_0}(\mathcal{A}^\delta_\alpha) - f_{\mathcal{A}_0}(\mathcal{A}^\delta_\alpha) - \langle \xi^\dagger, \mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger\rangle$, by Lemma 2 we have that,

$$\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\|^2 + \alpha D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) \le \delta^2 - \alpha\langle \xi^\dagger, \mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger\rangle = \delta^2 - \alpha(\langle \omega^\dagger, \mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger)\rangle + \langle \mathscr{E}, \mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger\rangle).$$

By Proposition 8, it follows that

$$\langle \omega^\dagger, \mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger)\rangle \le (1+\gamma)\|\omega^\dagger\|\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}(\mathcal{A}^\dagger)\| \le (1+\gamma)\|\omega^\dagger\|(\delta + \|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\|).$$

Therefore,

$$\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\|^2 + \alpha D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) \le \delta^2 + \alpha(1+\gamma)\|\omega^\dagger\|(\delta + \|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\|) + \alpha\|\mathscr{E}\| \cdot \|\mathcal{A}^\delta_\alpha - \mathcal{A}^\dagger\|.$$

Since $\|\mathscr{E}\|$ is arbitrarily small, it follows that, $(\zeta - \|\mathscr{E}\|)/\zeta > 0$. Moreover, since $f_{\mathcal{A}_0}$ is q-coercive with constant ζ.

For the case $q = 1$, the above inequalities imply that,

$$(\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\| - \alpha(1+\gamma)\|\omega^\dagger\|/2)^2 + \alpha(1 - 1/\zeta\|\mathscr{E}\|)D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) \le (\delta + \alpha(1+\gamma)\|\omega^\dagger\|)^2$$

Hence, the assertions follow.

For the case $q > 1$, we denote $\beta_1 = \|\mathscr{E}\|/\zeta$ and we have that,

$$\beta_1(D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger))^{1/q} \le \frac{\beta_1^q}{q} + \frac{1}{q}D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger).$$

Thus, assuming that $\beta_1 = \mathcal{O}(\delta^{1/q})$, we have that,

$$\left(\|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\| - \alpha\frac{1+\gamma}{2}\|\omega^\dagger\|\right)^2 + \alpha\frac{q-1}{q}D_{\xi^\dagger}(\mathcal{A}^\delta_\alpha, \mathcal{A}^\dagger) \le (\delta + \alpha(1+\gamma)\|\omega^\dagger\|)^2 + \alpha\frac{\beta_1^q}{q},$$

and the assertions follow. $\square$

The rates obtained in Theorem 5 can be seen as a measure of reliability of solutions, since it quantifies how better solutions become when the noise level decreases.

4 Morozov's Principle

We establish now a relaxed version of Morozov's discrepancy principle for the specific problem under consideration (see [17]). This is one of the most reliable ways of finding the regularization parameter α as a function of the data $\mathcal{U}^\delta$ and the noise level δ. Intuitively, the regularized solution should not fit the data more accurately than the noise level. We remark that it does not follow immediately because, the parameter now has to be chosen as a function of the noise level δ and the data $\mathcal{U}^\delta$. Thus, it is necessary to prove that such functional in fact satisfies the required criteria to achieve convergence and convergence rates when $\delta \to 0$ and $\mathcal{U}^\delta \to \widetilde{\mathcal{U}}$, where here $\widetilde{\mathcal{U}}$ represents the noiseless data.

From Equation (9), it follows that any $\mathcal{A} \in \mathfrak{Q}$ satisfying

$$\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\| \leq \delta \tag{13}$$

could be an approximate solution for (7). If $\mathcal{A}_\alpha^\delta$ is a minimizer of (10), then Morozov's discrepancy principle says that the regularization parameter α should be chosen from the condition

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| = \delta \tag{14}$$

when it is possible. In other words, the regularized solution should not satisfy the data more accurately than up to the noise level.

In what follows we consider a relaxed condition on Morozov's discrepancy principle, since the identity above is too restrictive. See [2].

During the following analysis, we also require that the functional $f_{\mathcal{A}_0}$ satisfies:

$$f_{\mathcal{A}_0}(\mathcal{A}) = 0 \iff \mathcal{A} = \mathcal{A}_0. \tag{15}$$

Fixing the noise level δ and the data $\mathcal{U}^\delta$, we define some auxiliary functionals and sets:

Definition 5. *[2] Define the functionals*

$$L : \mathcal{A} \in \mathfrak{Q} \longmapsto L(\mathcal{A}) = \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\| \in \mathbb{R}_+ \cup \{+\infty\}, \tag{16}$$

$$H : \mathcal{A} \in \mathfrak{Q} \longmapsto H(\mathcal{A}) = f_{\mathcal{A}_0}(\mathcal{A}) \in \mathbb{R}_+ \cup \{+\infty\}, \tag{17}$$

$$I : \alpha \in \mathbb{R}_+ \longmapsto I(\alpha) = \mathcal{F}_{\mathcal{A}_0,\alpha}^{\mathcal{U}^\delta}(\mathcal{A}_\alpha^\delta) \in \mathbb{R}_+ \cup \{+\infty\}. \tag{18}$$

We also define:

1. *The set of all minimizers of the functional (10) for each $\alpha \in (0, \infty)$*

$$M_\alpha := \left\{\mathcal{A}_\alpha^\delta \in \mathfrak{Q} : L(a_\alpha^\delta) \leq L(\mathcal{A}),\ \forall \mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D))\right\}.$$

   Note that we have extended $L(\mathcal{A})$ to be equal to $\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|$ when $\mathcal{A} \in \mathfrak{Q}$ and be equal to $+\infty$ otherwise.

2. *The set $\mathcal{L}$ of all $f_{\mathcal{A}_0}$-minimizing solutions of (8) (for this specific problem it is a unitary set).*

Definition 6. *For $1 < \tau_1 \leq \tau_2$ we choose $\alpha = \alpha(\delta, \mathcal{U}^\delta) > 0$ such that*

$$\tau_1 \delta \leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| \leq \tau_2 \delta \tag{19}$$

holds for some $\mathcal{A}_\alpha^\delta$ in M_α.

We start it by a classical Lemma:

Lemma 3. *[21, Lemma 2.6.1] The functional $H(\cdot)$ is non-increasing and the functionals $L(\cdot)$ and $I(\cdot)$ are non-decreasing with respect to $\alpha \in (0,\infty)$. In other words, for $0<\alpha<\beta$ we have*

$$\sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) \leq \inf_{\mathcal{A}_\beta^\delta \in M_\beta} L(\mathcal{A}_\beta^\delta), \quad \inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) \geq \inf_{\mathcal{A}_\beta^\delta \in M_\beta} H(\mathcal{A}_\beta^\delta) \quad \text{and } I(\alpha) \leq I(\beta). \tag{20}$$

Lemma 4. *The functional $I:(0,\infty)\to[0,\infty]$ is continuous. The sets*

$$M := \left\{ \alpha > 0 \,\middle|\, \inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) < \sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} L(\mathcal{A}_\alpha^\delta) \right\}$$

and

$$N := \left\{ \alpha > 0 \,\middle|\, \inf_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) < \sup_{\mathcal{A}_\alpha^\delta \in M_\alpha} H(\mathcal{A}_\alpha^\delta) \right\}$$

are at most countable and coincide. Moreover, the maps $L(\cdot)$ and $H(\cdot)$ are continuous in $(0,\infty)\backslash M$.

Proof: Let $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ be a convergent sequence converging to α^*. Thus, we can choose a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ where, for each $n\in\mathbb{N}$, $\mathcal{A}_n := \mathcal{A}_{\alpha_n}^\delta$, a minimizer of the functional defined in Equation (10) with α replaced by α_n. By the coerciveness of $f_{\mathcal{A}_0}$, $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is bounded and thus, it has a weakly convergent subsequence denoted by $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ with limit $\mathcal{A}^*$.
As $\mathcal{U}(\cdot)$ is weakly continuous, the norm of $L^2(0,S,W_2^{1,2}(D))$ is lower semi-continuous and $f_{a_0}(\cdot)$ is weakly lower semi-continuous, we have

$$\begin{aligned}
\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha^*}(\mathcal{A}^*) \;&\leq\; \liminf_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}_k) \;\leq\; \limsup_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}_k) \\
&\leq\; \lim_{k\to\infty} \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_k}(\mathcal{A}) \;=\; \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha^*}(\mathcal{A}), \forall\, \mathcal{A} \in \mathfrak{D},
\end{aligned}$$

which shows that $\mathcal{A}^* \in M_{\alpha^*}$. Thus, by the monotonicity of $I(\cdot)$ we have that $I(\cdot)$ is continuous at α^*.
If $\alpha\in M$ there are $\mathcal{A},\widetilde{\mathcal{A}} \in M_\alpha$ such that $L(\mathcal{A}) < L(\widetilde{\mathcal{A}})$, as $I(\mathcal{A}) = I(\widetilde{\mathcal{A}})$, we have

$$L(\mathcal{A}) \pm \alpha H(\mathcal{A}) < G(\widetilde{\mathcal{A}}) \pm \alpha H(\widetilde{\mathcal{A}}) \Leftrightarrow I(\mathcal{A}) - \alpha H(\mathcal{A}) < I(\widetilde{\mathcal{A}}) - \alpha H(\widetilde{\mathcal{A}}) \Leftrightarrow H(\mathcal{A}) > H(\widetilde{\mathcal{A}})$$

and $M\subset N$. The other inclusion is analogous. The countability of M follows by the fact that for each $\alpha\in M$ we can associate the interval $\mathcal{I}_\alpha = (\inf_{\mathcal{A}_\alpha^\delta\in M_\alpha} L(\mathcal{A}_\alpha^\delta), \sup_{\mathcal{A}_\alpha^\delta\in M_\alpha} L(\mathcal{A}_\alpha^\delta))$. By the monotonicity of L we have that for each $\alpha,\beta\in M$, $\mathcal{I}_\alpha\cap\mathcal{I}_\beta = \emptyset$. Therefore, we can define an injective map that associates each $\alpha\in M$ to an element of $\mathcal{I}_\alpha\cap\mathbb{Q}$.
The continuity of L and H with respect to α out of M follows by the same argument above about the continuity of I, with $\{\alpha_n\}_{n\in\mathbb{N}}$ and α^* in $(0,\infty)\backslash M$. □

The following lemma follows from the fact that the sets M_α are weakly closed.

Lemma 5. *For each $\overline{\alpha}>0$, there exist $\mathcal{A}_1,\mathcal{A}_2\in M_{\overline{\alpha}}$ such that*

$$L(\mathcal{A}_1) = \inf_{\mathcal{A}\in M_{\overline{\alpha}}} L(\mathcal{A}) \quad \text{and} \quad L(\mathcal{A}_2) = \sup_{\mathcal{A}\in M_{\overline{\alpha}}} L(\mathcal{A})$$

For the remaining part of this Section, we assume that $f_{\mathcal{A}_0}(\mathcal{A}_0) = 0$.

Proposition 9. *Let $1 < \tau_1 \le \tau_2$ be fixed. Suppose that $\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| > \tau_2\delta$. Then we can find $\underline{\alpha}, \overline{\alpha} > 0$, such that*

$$L(\mathcal{A}_1) < \tau_1\delta \le \tau_2\delta < L(\mathcal{A}_2),$$

where we denote $\mathcal{A}_1 := \mathcal{A}^\delta_{\underline{\alpha}}$ and $\mathcal{A}_2 := \mathcal{A}^\delta_{\overline{\alpha}}$.

Proof: First, let the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converge to 0. Then, we can find a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ with $\mathcal{A}_n \in M_{\alpha_n}$ for each $n \in \mathbb{N}$. Now, let $\mathcal{A}^\dagger$ be an $f_{\mathcal{A}_0}$-minimizing solution of (8). Hence, we have

$$L(\mathcal{A}_n)^2 \le I(\alpha_n) \le \mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_n}(\mathcal{A}^\dagger) \le \delta^2 + \alpha_n f_{\mathcal{A}_0}(\mathcal{A}^\dagger).$$

Thus, for n sufficiently large, we have $L(\mathcal{A}_n)^2 < (\tau_1\delta)^2$, as $\alpha_n f_{\mathcal{A}_0}(a^\dagger) \to 0$ and for this n we can set $\underline{\alpha} := \alpha_n$.

Now, we assume that $\alpha_n \to \infty$ and we take $\mathcal{A}_n$ as before, the

$$H(\mathcal{A}_n) \le \frac{1}{\alpha_n} I(\alpha_n) \le \frac{1}{\alpha_n}\mathcal{F}^{\mathcal{U}^\delta}_{\mathcal{A}_0,\alpha_n}(\mathcal{A}_0) = \frac{1}{\alpha_n}\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \to 0 \quad \textit{as } n \to \infty.$$

Thus, $\lim_{n\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_n) = 0$, which implies that $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ converges weakly to $\mathcal{A}_0$. Then, by the weak continuity of $\mathcal{U}(\cdot)$ and the lower semi-continuity of the norm, we have

$$\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \le \liminf_{n\to\infty} \|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}^\delta\|,$$

which shows the existence of $\overline{\alpha}$ such that $L(\mathcal{A}^\delta_{\overline{\alpha}}) > \tau_2\delta$. □

Now we require that, there is no $\alpha > 0$ with $\mathcal{A}_1, \mathcal{A}_2 \in M_\alpha$ such that

$$\|\mathcal{U}(\mathcal{A}_1) - \mathcal{U}^\delta\| < \tau_1\delta \le \tau_2\delta < \|\mathcal{U}(\mathcal{A}_2) - \mathcal{U}^\delta\|.$$

If such α exists, $\mathcal{A}_1$ would be a sufficiently good approximation for the solution of (8) given that $\|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^\delta\| \le \delta$. Thus, we can state the following theorem:

Theorem 6. *[2, Theorem 3.10] Under the above assumptions and the assumptions of Proposition 9, we have the existence of an $\alpha := \alpha(\delta) > 0$ and $\mathcal{A}^\delta_\alpha \in M_\alpha$ such that*

$$\tau_1\delta \le \|\mathcal{U}(\mathcal{A}^\delta_\alpha) - \mathcal{U}^\delta\| \le \tau_2\delta. \tag{21}$$

We now present the first main result of this section. It states that, if we use (19) to choose $\alpha = \alpha(\delta, \mathcal{U}^\delta)$, then the regularized solutions $\mathcal{A}^\delta_\alpha$ converges weakly to the $f_{\mathcal{A}_0}$-minimizing solution of (7).

Theorem 7. *Let $\delta > 0$ and $\mathcal{U}^\delta$ satisfy the hypotheses above. Then, the regularizing parameter $\alpha = \alpha(\delta, \mathcal{U}^\delta)$ obtained through Morozov's discrepancy principle (19) satisfies (11), i.e., the limits below*

$$\lim_{\delta\to 0+} \alpha(\delta, \mathcal{U}^\delta) = 0 \qquad \textit{and} \qquad \lim_{\delta\to 0+} \frac{\delta^2}{\alpha(\delta, \mathcal{U}^\delta)} = 0.$$

hold.

Proof: Take a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ in $(0, +\infty)$ converging to zero, fix the noiseless data $\widetilde{\mathcal{U}}$ and denote $\alpha_n := \alpha(\delta_n, \mathcal{U}^{\delta_n})$ the regularizing parameter chosen from Equation (19). Let $\mathcal{A}_n := \mathcal{A}^{\delta_n}_{\alpha_n}$ be a minimizer of (10) with δ_n, α_n and $\mathcal{U}^{\delta_n}$. This defines the sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$. As

$f_{\mathcal{A}_0}(\cdot)$ is coercive, such sequence has a weakly convergent subsequence, denoted by $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ with limit $\widetilde{\mathcal{A}}$.

By the weak lower semi-continuity of $\|\mathcal{U}(\cdot)-\widetilde{\mathcal{U}}\|$ and $f_{\mathcal{A}_0}$, we have the following estimates

$$\|\mathcal{U}(\widetilde{\mathcal{A}})-\widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k)-\widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k)-\mathcal{U}^{\delta_k}\|+\delta_k \leq \liminf_{k\to\infty}(\tau_2+1)\delta_k = 0, \quad (22)$$

i.e., $\widetilde{\mathcal{A}}$ is a solution for the inverse problem. From Equation (19), we have that

$$\lim_{k\to\infty} \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\| \leq \lim_{k\to\infty} \tau_2\delta_k = 0 \quad (23)$$

Therefore, for every $\mathcal{A}^\dagger \in \mathcal{L}$ we have the estimate

$$\tau_1^2\delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \delta_k + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \quad (24)$$

which shows that $\limsup_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. Then, it follows that

$$f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \leq \liminf_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq \limsup_{k\to\infty} f_{\mathcal{A}_0}(\mathcal{A}_k) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \quad (25)$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$. As $\mathcal{A}^\dagger$ is an $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem, we get that $\widetilde{\mathcal{A}} \in \mathcal{L}$ and $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. This implies that $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})$.

We now prove the first claim in the theorem. Assume that there exists an $\overline{\alpha} > 0$ and a subsequence $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $\alpha_k \geq \tilde{\alpha}$ for every $k \in \mathbb{N}$. As above select a sequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of minimizers of (10) with δ_k, α_k and $\mathcal{U}^{\delta_k}$. Define further the sequence $\{\overline{\mathcal{A}}_k\}_{k\in\mathbb{N}}$ of minimizers of (10) with δ_k, $\overline{\alpha}$ and $\mathcal{U}^{\delta_k}$.

As L in non-decreasing with respect to α and from Equation (19), it follows that

$$\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\| \leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}^{\delta_k}\| \leq \tau_2\delta_k \to 0 \quad (26)$$

On the other hand,

$$\begin{aligned} \limsup_{k\to\infty} \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) &\leq \limsup_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k)\right) \\ &\leq \limsup_{k\to\infty} \left(\|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}^\dagger)\right) = \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \end{aligned} \quad (27)$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$, as $\mathcal{U}(\mathcal{A}^\dagger) = \widetilde{\mathcal{U}}$, the noiseless data. By the coerciveness of $f_{\mathcal{A}_0}$, it follows that it has a convergent subsequence, denoted by $\{\overline{\mathcal{A}}_k\}_{k\in\mathbb{N}}$, with limit $\widetilde{\mathcal{A}} \in \mathfrak{Q}$. Thus, by (25), (26), the weak lower semi-continuity of $\|\mathcal{U}(\cdot) - \widetilde{\mathcal{U}}\|$ and $f_{\mathcal{A}_0}$, it follows that

$$\|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \|\mathcal{U}(\overline{\mathcal{A}}_k) - \widetilde{\mathcal{U}}\| \leq \liminf_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\| + \delta_k\right) = 0 \quad (28)$$

$$f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \leq \liminf_{k\to\infty} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \leq \limsup_{k\to\infty} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger), \quad (29)$$

for every $\mathcal{A}^\dagger \in \mathcal{L}$. As above, $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ and thus $f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. From the above estimates, we have

$$\begin{aligned} \|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) &\leq \liminf_{k\to\infty} \left(\|\mathcal{U}(\overline{\mathcal{A}}_k) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\overline{\mathcal{A}}_k)\right) \\ &\leq \liminf_{k\to\infty} \left(\|\mathcal{U}(\mathcal{A}) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A})\right) \\ &= \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^{\delta_k}\|^2 + \overline{\alpha} f_{\mathcal{A}_0}(\mathcal{A}), \end{aligned}$$

for every $\mathcal{A} \in \mathfrak{Q}$, i.e., $\widetilde{\mathcal{A}}$ is a minimizer for (10) with $\overline{\alpha}$ and the noiseless data $\widetilde{\mathcal{U}}$. Thus, by the convexity of $f_{\mathcal{A}_0}$ we have, for every $t \in [0,1)$,

$$f_{\mathcal{A}_0}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) \leq (1-t)f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) + tf_{\mathcal{A}_0}(\mathcal{A}_0) = (1-t)f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}).$$

On the other hand, from the above estimates it follows that

$$\begin{aligned}\overline{\alpha}f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) &= \|\mathcal{U}(\widetilde{\mathcal{A}}) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha}f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \\ &\leq \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha}f_{\mathcal{A}_0}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) \\ &\leq \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 + \overline{\alpha}(1-t)f_{\mathcal{A}_0}(\widetilde{\mathcal{A}})\end{aligned}$$

This implies that $\overline{\alpha}tf_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \leq \|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2$. As $\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}})$, by Proposition 5 with $\mathcal{H} = \mathcal{A}_0 - \mathcal{A}$, we have

$$\overline{\alpha}f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) \leq \lim_{t\to 0^+} \frac{1}{t}\|\mathcal{U}((1-t)\widetilde{\mathcal{A}} + t\mathcal{A}_0) - \widetilde{\mathcal{U}}\|^2 = 0$$

Therefore, $f_{\mathcal{A}_0}(\widetilde{\mathcal{A}}) = 0$ which, by hypothesis, can only hold if $\widetilde{\mathcal{A}} = \mathcal{A}_0$. However, by hypothesis, for every $\delta > 0$, $\mathcal{A}_0$ is chosen such that $\|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| \geq \tau_1\delta > \delta$. Thus,

$$\|\mathcal{U}(\mathcal{A}_0) - \widetilde{\mathcal{U}}\| \geq \|\mathcal{U}(\mathcal{A}_0) - \mathcal{U}^\delta\| - \|\mathcal{U}^\delta - \widetilde{\mathcal{U}}\| \geq (\tau_1 - 1)\delta > 0.$$

Therefore, we have achieved a contradiction with the fact that $\widetilde{\mathcal{U}} = \mathcal{U}(\widetilde{\mathcal{A}}) = \mathcal{U}(\mathcal{A}_0)$. We conclude that $\alpha(\delta,\mathcal{U}^\delta) \to 0$ when $\delta \to 0$.

Now for the second limit, take the subsequence $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of the sequence of the beginning of the proof, with $\delta_k \to 0$. We know that this sequence converges weakly to an $f_{\mathcal{A}_0}$-minimizing solution of the inverse problem $\mathcal{A}^\dagger$, with $f_{\mathcal{A}_0}(\mathcal{A}_k) \to f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$. Thus, from (19), it follows that

$$\begin{aligned}\tau_1^2\delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) &\leq \|\mathcal{U}(\mathcal{A}_k) - \mathcal{A}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}_k) \\ &\leq \|\mathcal{U}(\mathcal{A}^\dagger) - \mathcal{A}^{\delta_k}\|^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \\ &= \delta_k^2 + \alpha_k f_{\mathcal{A}_0}(\mathcal{A}^\dagger).\end{aligned}$$

This implies that

$$(\tau_1^2 - 1)\frac{\delta^2}{\alpha_n} \leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger) - f_{\mathcal{A}_0}(\mathcal{A}_k) \to 0.$$

□

We state now that under the Morozov's choice for the regularization parameter α we achieve the same convergence rates of the Theorem 5. This is the second main result of this section.

Theorem 8. *Assume that $\mathcal{A}_\alpha^\delta$ is a minimizer of (10)and $\alpha = \alpha(\delta,\mathcal{U}^\delta)$ is chosen by Morozov's discrepancy principle (19). Then, Theorem 5 holds, i.e., we have the following estimates*

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| = \mathcal{O}(\delta) \quad \text{and} \quad D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta, \mathcal{A}^\dagger) = \mathcal{O}(\delta), \tag{30}$$

with $\xi^\dagger \in \partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$.

Proof: Assume that $\mathcal{A}^\dagger$ is an $f_{\mathcal{A}_0}$-minimizing solution of (7) and $\mathcal{A}_\alpha^\delta \in M_\alpha$. The first estimate is trivial as

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}(\mathcal{A}^\dagger)\| \leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta) - \mathcal{U}^\delta\| + \|\mathcal{U}^\delta - \mathcal{U}(\mathcal{A}^\dagger)\| \leq (\tau_2 + 1)\delta.$$

By (19) it follows that

$$\begin{aligned}\tau_1\delta^2+\alpha f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta) &\leq \|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}^\delta\|^2+\alpha f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta)\\ &\leq \|\mathcal{U}(\mathcal{A}^\dagger)-\mathcal{U}^\delta\|^2+\alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger) \quad \leq \quad \delta^2+\alpha f_{\mathcal{A}_0}(\mathcal{A}^\dagger),\end{aligned}$$

implying that $f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta)\leq f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ as $\tau_1-1>0$. Hence, by the definition of Bregman distance we have, for every $\xi^\dagger\in\partial f_{\mathcal{A}_0}(\mathcal{A}^\dagger)$ satisfying the source condition,

$$\begin{aligned}D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger) &= f_{\mathcal{A}_0}(\mathcal{A}_\alpha^\delta)-f_{\mathcal{A}_0}(\mathcal{A}^\dagger)-\langle\xi^\dagger,\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\rangle\\ &\leq |\langle\xi^\dagger,\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\rangle| = |\langle\mathcal{U}'(\mathcal{A}^\dagger)^*\omega^\dagger+\mathcal{E},\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\rangle|\\ &\leq \|\omega^\dagger\|\|\mathcal{U}'(\mathcal{A}^\dagger)(\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger)\|+\|\mathcal{E}\|\|\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\|\\ &\leq (1+\gamma)\|\omega^\dagger\|\|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}(\mathcal{A}^\dagger)\|+\frac{1}{\zeta}\|\mathcal{E}\|D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger)\end{aligned} \tag{31}$$

The last inequality follows by the tangential cone condition and the 1-coerciveness with constant ζ of $f_{\mathcal{A}_0}$. As $\xi^\dagger$ can be chosen with $\|\mathcal{E}\|$ arbitrarily small, it follows that

$$1-\frac{1}{\zeta}\|\mathcal{E}\|>0$$

and then, by (19)

$$D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger)\leq\frac{\zeta}{\zeta-\|\mathcal{E}\|}(1+\gamma)\|\omega^\dagger\|\|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}(\mathcal{A}^\dagger)\|\leq\tau_2\frac{\zeta}{\zeta-\|\mathcal{E}\|}(1+\gamma)\|\omega^\dagger\|\cdot\delta.$$

□

Remark 2. *For $f_{\mathcal{A}_0}$ q-coercive with $q>1$, a reasoning as the one used in Equation (31), gives that*

$$\begin{aligned}D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger) &\leq \beta_1(D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger))^{1/q}+\beta_2\|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}(\mathcal{A}^\dagger)\|\\ &\leq \frac{\beta_1^q}{q}+\frac{1}{q}D_{\xi^\dagger}(\mathcal{A}_\alpha^\delta,\alpha)+\beta_2\|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}(\mathcal{A}^\dagger)\|.\end{aligned}$$

Assume further that $\beta_1=\mathcal{O}(\delta^{\frac{1}{q}})$. As

$$\|\mathcal{U}(\mathcal{A}_\alpha^\delta)-\mathcal{U}(\mathcal{A}^\dagger)\|=\mathcal{O}(\delta)$$

it follows that

$$\|\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\|^q\leq\frac{1}{\zeta}D_\xi(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger)=\mathcal{O}(\delta).$$

One interesting example is the quadratic functional

$$f_{\mathcal{A}_0}(\mathcal{A})=\|\mathcal{A}-\mathcal{A}_0\|^2_{H^l(0,S,H^{1+\varepsilon}(D))},$$

which is 2-coercive with constant 1, since its Bregman distance is

$$D_{2(\widetilde{\mathcal{A}}-\mathcal{A}_0)}(\mathcal{A},\widetilde{\mathcal{A}})=\|\mathcal{A}-\widetilde{\mathcal{A}}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

Thus, by Theorem 5 it follows that

$$\mathcal{O}(\delta)=D_{2(\mathcal{A}^\dagger-\mathcal{A}_0)}(\mathcal{A}_\alpha^\delta,\mathcal{A}^\dagger)=\|\mathcal{A}_\alpha^\delta-\mathcal{A}^\dagger\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

5 Numerical Results

We first performed tests with synthetic data in order to test accuracy and advantages of the methods. Then, we present some reconstructions with real data.

We note that Problem (2) is solved by a Crank-Nicolson scheme [22]. Since we shall use a gradient method to solve numerically the inverse problem, we define the quadratic residual $J^\delta(\mathcal{A})$ and its gradient $\nabla J^\delta(\mathcal{A})$:

$$J^\delta(\mathcal{A}) := \|\mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta\|^2_{L^2(0,S,L^2(D))} = \int_0^S \|F(s,a(s)) - u^\delta(s)\|^2_{L^2(D)} ds, \tag{32}$$

$$\begin{aligned} \langle \nabla J^\delta(\mathcal{A}), \mathcal{H}\rangle_{H^l(0,S,H^{1+\varepsilon}(D))} &= 2\langle \mathcal{U}(\mathcal{A}) - \mathcal{U}^\delta, \mathcal{U}'(\mathcal{A})\mathcal{H}\rangle_{L^2(0,S,L^2(D))} \\ &= 2\int_0^S \int_D \{[v(u_{yy} - u_y)h(t)](s,a(s))\}(\tau,y)d\tau dy ds, \end{aligned} \tag{33}$$

where, for each $s \in [0,S]$, v is the solution of the adjoint equation,

$$v_\tau + (av)_{yy} + (av)_y + bv_y = u(t,a) - u^\delta(s) \tag{34}$$

with homogeneous boundary condition. Furthermore, $\{V : s \mapsto v(s)\} \in L^2(0,S,W^{1,2}_2(D))$. We solve Problem (34) numerically also by a Crank-Nicolson scheme.

For simplicity, in the following examples we assume that the index l of $H^l(0,S,H^{1+\varepsilon}(D))$ is equal to 1 and considered the standard regularization functional

$$f_{\mathcal{A}_0}(\mathcal{A}) = \|\mathcal{A} - \mathcal{A}_0\|^2_{H^l(0,S,H^{1+\varepsilon}(D))}.$$

5.1 Examples with Synthetic Data

Consider the following local volatility surface:

$$a(s,u,x) = \begin{cases} 0.4(1 - 0.4\mathrm{e}^{-0.5(u-s)})\cos(1.25\pi\log(x/s)), & \text{if } (u,x) \in (0,1.25] \times [-0.4,0.4], \\ 0.4, & \text{otherwise.} \end{cases} \tag{35}$$

The data is generated in a fine mesh and then it is collected in a coarser grid with an additive noise. We assume that such noise term is determined by a zero-mean Gaussian pseudo-random variable with variance equals to the square of the maximum value reached by the noiseless data. We proceed in this way, in order to avoid a so-called inverse crime [20].

In order to perform the numerical tests, we assume that, $r = 0.03$, the log-moneyness domain is $[-5,5]$ and the time to maturity domain is $[0,1]$. Moreover, the time and log-moneyness step sizes are $\Delta t = 0.004$ and $\Delta y = 0.1$, respectively. We also assume that $s \in [29.5, 32.5]$ with different step sizes, $\Delta s = 0.25, 0.1, 0.01$.

In what follows, we refer to standard Tikhonov as the case when we consider a single price surface. We use the terminology "online" Tikhonov whenever we use more than one single price surface.

Figure 1 illustrates that, when the noise level decreases by increasing the accuracy of data, the resulting reconstructions become more similar to the original local volatility surface, as stated by Theorems 4, 5, 7 and 8.

In Figure 2, we show that the online Tikhonov presents better solutions than the standard one, as we increase the number of price surfaces. Observe that, as it was mentioned

above, the parameters are considered in $H^l(0, S, H^{1+\varepsilon}(D))$ with $l = 1$. Here, the regularization parameter was obtained by the Morozov's discrepancy principle and the regularization functional is the standard one.

Figure 3 shows how the $L^2(D)$ distance between the reconstructions and the original local variance behave as a function of the number of price surface: it is constant for standard Tikhonov and non-increasing for online Tikhonov.

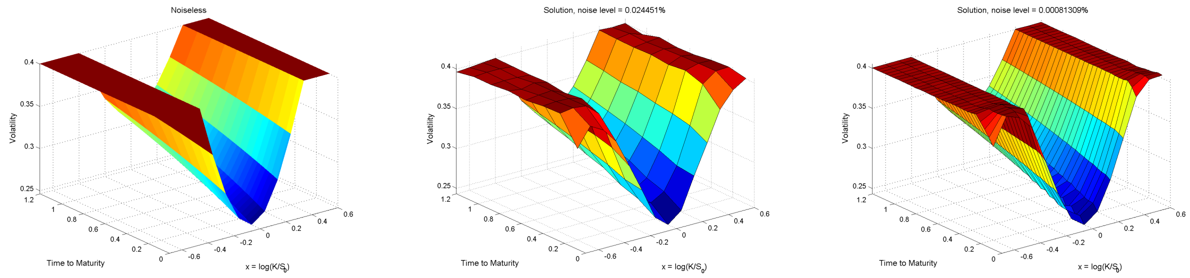

Figure 1: Original local volatility surface (image 1) and reconstructions with noise level $\delta = 0.035$ (image 2) and $\delta = 0.01$ (image 3), respectively. As the noise level decreases, the reconstructions become more accurate.

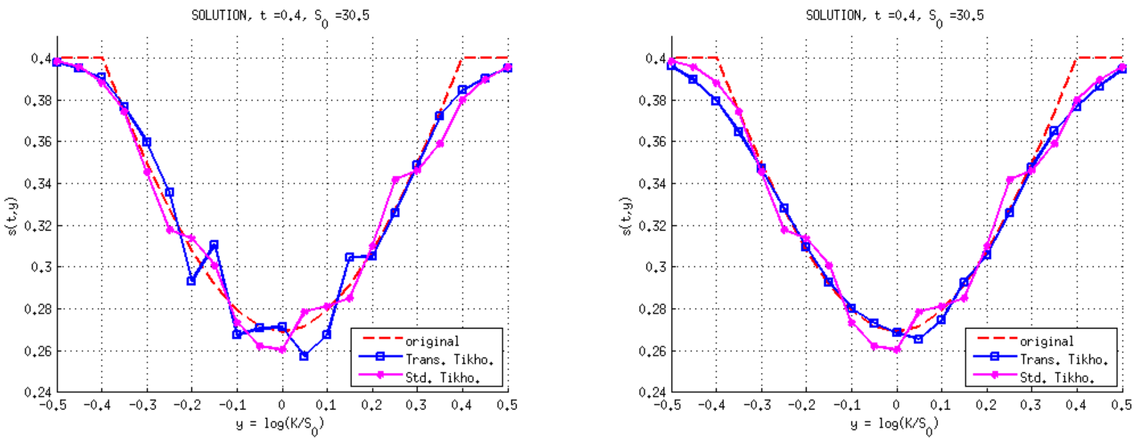

Figure 2: Comparison between standard and online Tikhonov. As the number of price surfaces increases, the reconstructions become more accurate.

5.2 Market Data

We present now some reconstructions of local volatility by online Tikhonov regularization from real data. We make the same assumptions of the previous section about the parameter space. We consider seven price surfaces in each experiment. The data corresponds to vanilla option prices on futures of WTI oil and Henry Hub natural gas [13].

Note that, in order to use the framework developed in the previous sections, we assumed that, local volatility is indexed by the unobservable spot price, instead of the future price. For more details on such examples, see Chapters 4 and 5 of [1].

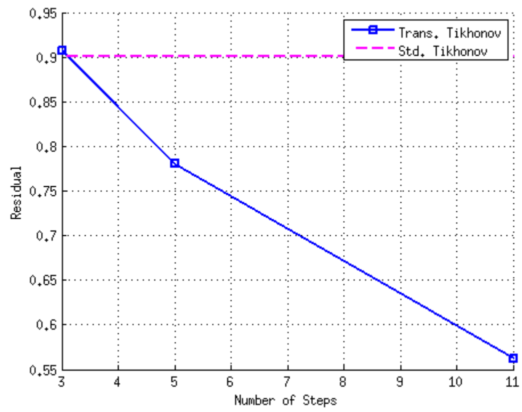

Figure 3: L^2 distance between original local variance and its reconstructions, as a function of the number of price surfaces. it is constant for standard Tikhonov and non-increasing for on line Tikhonov.

Figures 4 and 5 present such reconstructions for WTI and HH, respectively. We collected the data prices for Henry Hub natural gas and WTI oil between 2011/11/16 and 2011/11/25, i.e., seven consecutive commercial days.

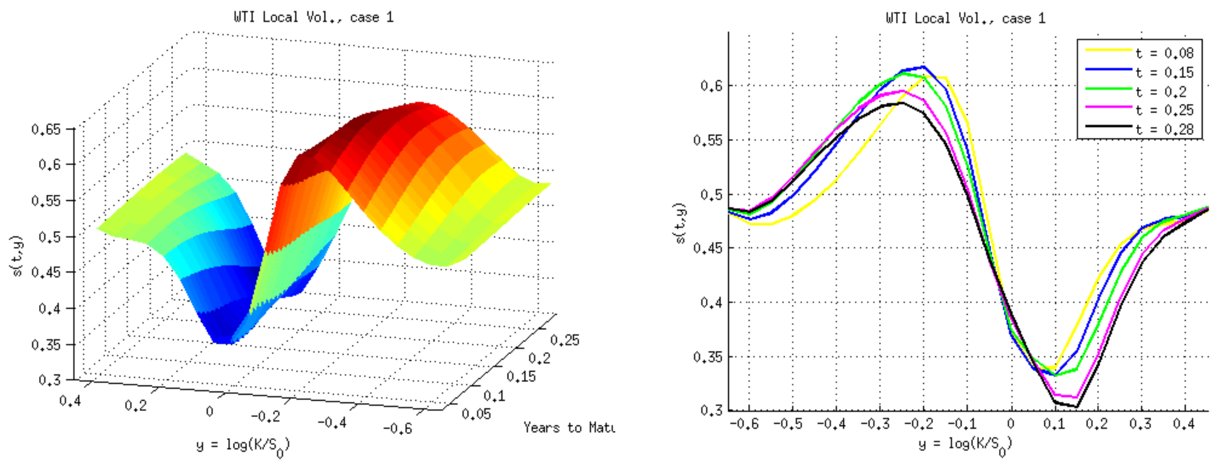

Figure 4: Local Volatility reconstruction from European vanilla options on futures of WTI oil. We used online Tikhonov regularization with the standard functional.

6 Conclusions

In this paper we have used convex regularization tools to solve the inverse problem associated to Dupire's local volatility model when there is a steady flow of data. We first established results concerning existence, stability and convergence of regularized solutions, as consequences of convex regularization tools and the regularity of the forward operator. We also proved some convergence rates. Furthermore, we established a Morozov's discrepancy principle under a general framework, following [2]. Such analysis allowed us to implement the algorithms and perform numerical tests.

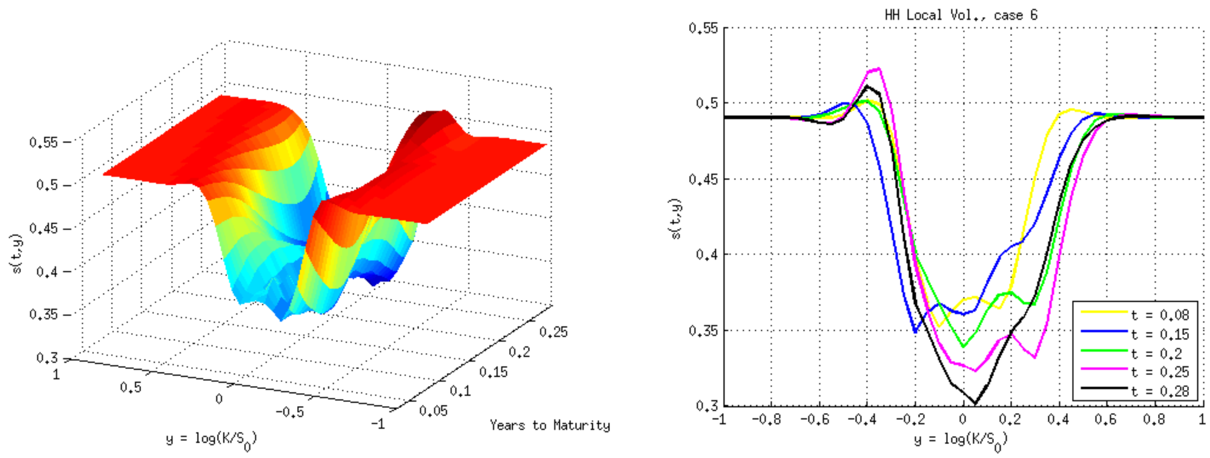

Figure 5: Local Volatility reconstruction from European vanilla options on futures of Henry Hub natural gas. We used online Tikhonov regularization with the standard functional.

The main contribution, *vis a vis* previous works, and in particular [6], is that we extended the convex regularization techniques to incorporate the information and data stream that is constantly supplied by the market. Furthermore, we have proved a Morozov's discrepancy principle that is suitable to this context.

A natural extension of the current work is the application of these techniques to the context of future markets, where the underlying asset is the future price of some financial instrument or commodity. In such markets, vanilla options represent a key instrument in hedging strategies of companies and in general they are far more liquid than in equity markets. The warning here is that, in general, we do not have an entire price surface. Actually in this case, we have only an option price curve for each future's maturity. Thus, in order to apply the techniques above to this context, it is necessary to assembly all option prices for futures on the same instrument (financial or commodity) in a unique surface in an appropriate way. This was discussed in [1, Chapter 4] and will be published elsewhere.

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A Proofs, Technical Results and Definitions

In this appendix we collect technical results and definitions that were used in the remaining parts of the article. We also present the proofs of some results of from Section 2.

A.1 Bregman Distance and q-Coerciveness

Definition 7. *[19, Definition 3.15] Let X denote a Banach space and $f : D(f) \subset X \to \mathbb{R} \cup \infty$ be a convex functional with sub-differential $\partial f(x)$ in $x \in D(f)$. The Bregman distance (or divergence) of f at $x \in D(f)$ and $\xi \in \partial f(x) \subset X^*$ is defined by*

$$D_\xi(\tilde{x}, x) = f(\tilde{x}) - f(x) - \langle \xi, \tilde{x} - x \rangle, \tag{36}$$

for every $\tilde{x} \in X$, with $\langle \cdot, \cdot\cdot \rangle$ the dual product of X^* and X. Moreover, the set

$$\mathcal{D}_B(f) = \{x \in D(f) \ : \ \partial f(x) \neq \emptyset\}$$

is called the Bregman domain of f.

We stress that the Bregman domain $\mathcal{D}_B(f)$ is dense in $D(f)$ and the interior of $D(f)$ is a subset of $\mathcal{D}_B(f)$. The map $\tilde{x} \mapsto D_\xi(\tilde{x}, x)$ is convex, non-negative and satisfies $D_\xi(x, x) = 0$. In addition, if f is strictly convex, then $D_\xi(\tilde{x}, x) = 0$ if and only if $\tilde{x} = x$. For a survey in Bregman distances see [3, Chapter I].

Definition 8. *For $1 \leq q < \infty$ and $x \in D(f)$, the Bregman distance $D_\xi(\cdot, x)$ is said to be q-coercive with constant $\zeta > 0$ if*

$$D_\xi(y, x) \geq \zeta \|y - x\|_X^q$$

for every $y \in D(f)$.

A.2 Equi-Continuity

Let X and Y be locally convex spaces. Fix the sets $B_X \subset X$ and $M \subset C(B_X, Y)$. A set M is called equi-continuous on B_X if for every $x_0 \in B_X$ and every zero neighborhood, $V \subset Y$ there is a zero neighborhood $U \subset X$ such that $G(x_0) - G(x) \in V$ for all $G \in M$ and all $x \in B_X$ with $x - x_0 \in U$. Furthermore, M is called uniformly equi-continuous if for every zero neighborhood $V \subset Y$ there exists a zero neighborhood $U \subset X$ such that $G(x) - G(x') \in V$ for all $G \in M$ and all $x, x' \in B_X$ with $x - x' \in U$.

From [14] we have the technical result:

Proposition 10. *Let $F : [0,T] \times B_X \longrightarrow Y$ be a function, and B_X, X and Y be as above. If $M_1 := \{F(t,\cdot) : t \in [0,T]\} \subset C(B_X, Y)$, $M_2 := \{F(\cdot, x) : x \in B_X\} \subset C([0,T], Y)$ and M_1 (respectively M_2) is equi-continuous, then F is continuous. Reciprocally, if F is continuous, then M_1 is equi-continuous and if additionally B_X is compact, then M_2 is equi-continuous, too.*

A.3 Proof of Results from Section 2

Proof of Theorem 1: *Well Posedness:* Take an arbitrary $\widetilde{\mathcal{A}} \in \mathfrak{Q}$, by the continuity of $\widetilde{\mathcal{A}}$ (see Proposition 1) and F, it follows that $t \mapsto F(s, \tilde{a}(s))$ is continuous and then weakly measurable. Therefore, $s \mapsto \|F(s, a(s))\|_{W_2^{1,2}(D)}$ is bounded, then $\mathcal{U}(\widetilde{\mathcal{A}}) \in L^2(0, S, W_2^{1,2}(D))$, which asserts the well-posedness of $\mathcal{U}(\cdot)$.
Continuity: As $F : [0,S] \times Q \longrightarrow W_2^{1,2}(D)$ is continuous, it follows by Proposition 1 that the set $\{F(s,\cdot) \mid s \in [0,S]\} \subset C(Q, W_2^{1,2}(D))$ is uniformly equi-continuous, i.e., given $\epsilon > 0$, there is a $\delta > 0$ such that, for all $a, \tilde{a} \in Q$ satisfying $\|a - \tilde{a}\| < \delta$, we have that

$$\sup_{s\in[0,S]} \|F(s,a) - F(s,\tilde{a})\| < \epsilon.$$

Thus, given $\epsilon > 0$ and $\mathcal{A}, \widetilde{\mathcal{A}} \in \mathfrak{Q}$ such that $\sup_{s\in[0,S]} \|a(s) - \tilde{a}(s)\|_{H^{1+\varepsilon}(D)} < \delta$, then, by the uniform equi-continuity of $\{F(s,\cdot), s \in [0,S]\}$, it follows that

$$\|\mathcal{U}(\mathcal{A}) - \mathcal{U}(\widetilde{\mathcal{A}})\|^2_{L^2(0,S,W_2^{1,2}(D))} = \int_0^S \|F(s,a(s)) - F(s,\tilde{a}(s))\|^2_{W_2^{1,2}(D)} ds < \epsilon^2 \cdot S,$$

which asserts the continuity of $\mathcal{U}(\cdot)$.
Compactness: It is sufficient to prove that, given an $\epsilon > 0$ and a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ converging weakly to $\widetilde{\mathcal{A}}$, it follows that there exist an n_0 and a weak zero neighborhood U of $H^l(0, S, H^{1+\varepsilon}(D))$ such that for $n > n_0$, $\mathcal{A}_n - \widetilde{\mathcal{A}} \in U$ and

$$\|\mathcal{U}(\mathcal{A}_n) - \mathcal{U}(\widetilde{\mathcal{A}})\|_{L^2(0,S,W_2^{1,2}(D))} < \epsilon.$$

Following the same arguments of the proof of Lemma 1, we can find a set of functionals $\mathcal{C}_{n,m} \in H^l(0, S, H^{1+\varepsilon}(D))^*$, defining such zero neighborhood U. We first note that, since F is weak continuous, it follows that, given an $\epsilon > 0$, there are $\alpha_1, ..., \alpha_N \in H^{1+\varepsilon}(D)$ and $\delta > 0$, such that

$$\sup_{s\in[0,S]} \|F(s,a) - F(s,\tilde{a})\| < \epsilon/S$$

for all $a, \tilde{a} \in B$ with

$$\max\{|\langle a - \tilde{a}, \alpha_n\rangle_{H^{1+\varepsilon}(D)}| \,|n = 1, ..., N\} < \delta. \tag{37}$$

By Proposition 1 it follows that $\langle \mathcal{A}, \alpha_n\rangle_{H^{1+\varepsilon}(D)} \in H^l([0,S])$ with its norm bounded by $\|\mathcal{A}\|_l \|\alpha_n\|_{H^{1+\varepsilon}(D)}$. Then, there is a closed and bounded ball

$$A \subset H^l([0,S]) \text{ containing } \langle \mathcal{A}, \alpha_n\rangle_{H^{1+\varepsilon}(D)}, \text{ for all } n = 1, ..., N,$$

and $\mathcal{A} \in \mathbb{B}$. For $n = 1, ..., N$ and the same $\delta > 0$ of (37), there are $f_{n,1}, ..., f_{n,M(n)} \in H^l([0,S])$ and $\xi_n > 0$ such that, $\|f\|_{C([0,S])} < \delta$ for every $f \in A$ satisfying

$$\max\{|\langle f, \alpha_n\rangle_{H^{1+\varepsilon}(D)} \mid m = 1, ..., M(n)\} < \xi_n.$$

Now, define $\mathcal{C}_{n,m} := \alpha_n \otimes f_{n,m}$, with $n = 1, ..., N$ and $m = 1, ..., M(n)$. It is an element of $H^l(0, S, H^{1+\varepsilon}(D))^*$, where, for each $\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D))$, we have that

$$\langle \mathcal{A}, \mathcal{C}_{n,m} \rangle_l = \langle \langle \mathcal{A}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}, f_{n,m} \rangle_{H^l([0,S])} = \sum_{k \in \mathbb{Z}} (1 + |k|^l)^2 \langle \hat{a}(k), \alpha_n \rangle_{H^{1+\varepsilon}(D)} \hat{f}_{n,m}(k).$$

These functionals define a weak zero neighborhood $U := \cap_{n=1}^N U_n$ with

$$U_n := \{\mathcal{A} \in H^l(0, S, H^{1+\varepsilon}(D)) : |\langle \mathcal{A}, \mathcal{C}_{n,m} \rangle_l| < \xi_n, \ m = 1, ..., M(n)\}.$$

Therefore, if $\{\mathcal{A}_k\}_{k \in \mathbb{N}} \subset \mathbb{B}$ converges weakly to $\widetilde{\mathcal{A}} \in \mathbb{B}$, then for a sufficient large k, $\mathcal{A}_k - \widetilde{\mathcal{A}} \in U$ and by the definition of U, we have that for each $n = 1, ..., N$,

$$\xi_n > |\langle \mathcal{A} - \widetilde{\mathcal{A}}, \mathcal{C}_{n,m} \rangle_l| = |\langle \langle \mathcal{A} - \widetilde{\mathcal{A}}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}, f_{n,m} \rangle_{H^l([0,S])}|$$

for all $m = 1, ..., M(n)$. By the choice of the $f_{n,m} \in H^l([0, S])$, it follows that

$$\|\langle \mathcal{A}_k - \widetilde{\mathcal{A}}, \alpha_n \rangle_{H^{1+\varepsilon}(D)}\|_{H^l([0,S])} < \delta \text{ for all } n = 1, ..., N,$$

which implies that $\|\mathcal{U}(\mathcal{A}_k) - \mathcal{U}(\widetilde{\mathcal{A}})\|_{L^2(0,S,W_2^{1,2}(D))} \leq \epsilon \cdot T$.
Weak Continuity: The weak continuity follows directly from the proof of compactness, as we use the same framework, only changing the compactness of F, by the weakly equi-continuity of $\{F(s, \cdot) : \ s \in [0, S]\}$ on bounded subsets of Q.
Weak Closedness: It is true since $\mathfrak{Q}$ is weakly closed and $\mathcal{U}(\cdot)$ is weakly continuous. $\square$

Proof of Proposition 5 By Proposition 4, the family of operators $\{F(s, \cdot) \ : \ s \in [0, S]\}$ is Frechét equi-differentiable. Take $\widetilde{\mathcal{A}}, \mathcal{H} \in H^l(0, S, H^{1+\varepsilon}(D))$, such that $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H} \in \mathfrak{Q}$. Thus, define the one sided derivative of $\mathcal{U}(\cdot)$ at $\widetilde{\mathcal{A}}$ in the direction $\mathcal{H}$ as

$$\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H} := \{s \mapsto \partial_a F(s, \tilde{a}(s))h(s)\},$$

where for each $s \in [0, S]$, dropping t to easy the notation, $\partial_a F(s, \tilde{a})h$ is the solution of

$$-v_\tau + a(v_{yy} - v_y) + bv_y = h(u_{yy} - u_y)$$

with homogeneous boundary conditions and $u = u(s, a(s))$. From Proposition 3 we have the estimate

$$\|\partial_a F(s, \tilde{a}(s))h(s)\|_{W_2^{1,2}(D)} \leq C\|h(s)\|_{L^2(D)}\|u_{yy}(s, \tilde{a}(s)) - u_y(s, \tilde{a}(s))\|_{L^2(D)}.$$

Where $\|u_{yy}(s, a) - u_y(s, a)\|_{L^2(D)}$ is uniformly bounded in $[0, S] \times Q$. Thus, $\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}$ is well defined and

$$\begin{aligned} \left\|\mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{H}\right\|^2_{L^2(0,S,W_2^{1,2}(D))} &= \int_0^S \|\partial_a F(s, \tilde{a}(s))h(s)\|^2_{W_2^{1,2}(D)} ds \\ &\leq C \int_0^S \|h(s)\|_{L^2(D)}\|u_{yy}(s, \tilde{a}(s)) - u_y(s, \tilde{a}(s))\|_{L^2(D)} ds \\ &\leq c \int_0^S \|h(s)\|^2_{L^2(D)} ds = c\|\mathcal{H}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))} \end{aligned}$$

Therefore, $\mathcal{U}'(\widetilde{\mathcal{A}})$ can be extended to a bounded linear operator from $H^l(0, S, H^{1+\varepsilon}(D))$ to $L^2(0, S, W_2^{1,2}(D))$.

Now, let $\widetilde{\mathcal{A}}, \mathcal{H}, \mathcal{G} \in H^l(0, S, H^{1+\varepsilon}(D))$ be such that,

$$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}} + \mathcal{H}, \widetilde{\mathcal{A}} + \mathcal{G}, \widetilde{\mathcal{A}} + \mathcal{H} + \mathcal{G} \in Q.$$

Denote, $v := u(s, a(s) + h(s)) - u(s, a(s))$. Thus,

$$w := (\partial_a u(s, a(s) + h(s))g(s) - \partial_a u(s, a(s))g(s))$$

satisfies

$$-w_\tau + a(w_{yy} - w_y) = -q[v_{yy} - v_y] - h[(\partial_a u(s, a + h)g)_{yy} - (\partial_a u(s, a + h)g)_y],$$

with homogeneous boundary conditions (dropping the dependence on s). As above, we have

$$\begin{aligned}
\left\| \mathcal{U}'(\widetilde{\mathcal{A}} + \mathcal{H})\mathcal{G} - \mathcal{U}'(\widetilde{\mathcal{A}})\mathcal{G} \right\|^2_{L^2(0,S,W_2^{1,2}(D))} &= \int_0^S \|w\|^2_{W_2^{1,2}(D)} ds \\
&\le c_1 \int_0^S \|q(s)\|^2_{L^2(D)} \|v_{yy}(s, \tilde{a}(s)) - v_y(s, \tilde{a}(s))\|^2_{L^2(D)} ds \\
&+ c_2 \int_0^S \|h(s)\|^2_{L^2(D)} \|\partial_a u(s, a(s) + h(s))g(s)\|^2_{W_2^{1,2}(D)} ds \\
&\le C \|\mathcal{H}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))} \|\mathcal{G}\|^2_{H^l(0,S,H^{1+\varepsilon}(D))},
\end{aligned}$$

which yields the Lipschitz condition. $\square$